I YEAR – I SEMESTER
COURSE CODE: 7MMA1C2

CORE COURSE-II – ANALYSIS – I

Unit I

Basic Topology: Metric Spaces – Compact sets – Perfect sets – Connected sets.

Unit II

Numerical sequences and series; Convergent sequences, Subsequences, Cauchy sequences, Upper and Lower limits – Special sequences, Series, Series of non–negative terms. The number e – The root and ratio tests.

Unit III

Power series – Summation by parts – Absolute convergence – Addition and Multiplication of series – Rearrangements

Unit IV

Continuity: Limits of functions – Continuous functions, Continuity and Compactness, Continuity and Connectedness – Discontinuities – Monotonic functions – infinite limits and limits at infinity.

Unit V

Differentiation: The derivative of a real function – Mean value theorems – the continuity of derivatives – L'Hospital's rule – Derivatives of Higher order – Taylor's theorem Differentiation of vector – valued functions.

Text Book

Walter Rudin, Principles of Mathematical Analysis, III Edition (Relevant portions of chapters II, III, IV & V), McGraw-Hill Book Company, 1976.

Books for Supplementary Reading and Reference:

1. H.L.Royden, Real Analysis, Macmillan Publ.co., Inc. 4th edition, New York, 1993.
2. V.Ganapathy Iyer, Mathematical Analysis, Tata McGraw Hill, New Delhi, 1970.
3. T.M.Apostal, Mathematical Analysis, Narosa Publ. House, New Delhi, 1985.

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$\therefore$ constructs a sequence $\{n_k\}$ as follows

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Choose $n_1, n_2, \ldots, n_{k-1}$ $(k=2,3,4,\ldots)$

Let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$

Put $f(k) = x_{n_k}$ $(k=1,2,3,\ldots)$ we get a 1-1 correspondence between E and J

$\therefore$ Hence E is countable

Hence the proof.

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Definition

Let $\{E_\alpha\}$ be the collection of sets then the union of the sets E_α is defined by the set 'S'

$$S = \{x : x \in E_\alpha \text{ for atleast one } \alpha\}$$

$$S = \bigcup_{\alpha \in A} E_\alpha$$

Note:-

If $x \in A$ where A is the set of all positive integer then $S = \bigcup_{m=1}^{\infty} E_m$

If $A = \{1, 2, \ldots n\}$ then

$$S = \bigcup_{m=1}^{n} E_m \quad \text{(or)} \quad S = E_1 \cup E_2 \cup \ldots \cup E_n$$

The intersection of the set E_α :-

Let $\{E_\alpha\}$ be the collection of the sets then the intersection of the sets E_α is defined by the set P

$$P = \{x : x \in E_\alpha \text{ For every } \alpha\}$$

$$P = \bigcap_{\alpha \in A} E_\alpha$$

Note:- If $A \cap B$ is non-empty A and B intersects otherwise they are disjoint

Example:- 1) Let $E_1 = \{1, 2, 3\}$ and $E_2 = \{2, 3, 4\}$

Theorem 1·3

Let A be a countable set and let B_n be the set of all n $(a_1, a_2, \dots a_n)$ where $a_k \in A$ $(k = 1, 2, 3, \dots n)$ and the element $a_1, a_2 \dots a_n$ need not be distinct then B_n is countable

Proof

We prove this theorem by induction method

If $n=1$

Clearly, $A=B$ then B is countable

Assume that B_{n-1} is countable $(n = 2, 3, \dots)$

Claim: B_n is countable

The elements of B_n are of the form (b, a) where $b \in B_{n-1}$ and $a \in A$. For every fixed b,

The set of pairs (b, a) is equivalent to A

Hence (b, a) is countable

Thus B_n is the union of a countable sets

B_n is countable (By thm 1·2)

Hence proved

Corollary

The set of all rational numbers is countable

Proof:

Since every rational number is of the form b/a where a and b are integers

We consider it by the set of pairs (a, b)

$\therefore B_2$ is countable (By thm 1·3)

$\therefore$ The set of fractions b/a is countable

Hence proved

Theorem: 1·4

Let A be the set of all sequence whose elements are the digit 0 and 1 then A is uncountable

9. If X is a metric space and $E \subset X$ then a) $\bar{E}$ is closed b) $E = \bar{E}$ iff E is closed (c) $\bar{E} \subset F$ for any closed set $F \subset X$ such that $E \subset F$

Proof: a) Let X be a metric space and $E \subset X$

we claim $\bar{E}$ is closed

Let $p \in X$ and $p \notin \bar{E}$ 15

then p is neither a point of E nor a limit point E $\therefore$ p has a neighbourhood

which does not intersect E

$\therefore$ The complement of $\bar{E}$ is open

By theorem,

Thus the set E is open iff its complemen is closed $\therefore$ $\bar{E}$ is closed

b) suppose $E = \bar{E}$

we claim:- E is closed

By (a) $\bar{E}$ is closed

Conversely,

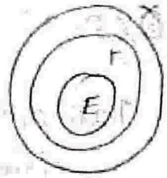
suppose E is closed

then every limit point of E is a point of E and by the definition of closure $\bar{E} = E \cup E'$

Where E' is the set of all limit point of E in X

$\therefore E = \bar{E}$

C) suppose F is closed



Let $E \subset F$

then $F' \subset F$ [F is closed]

$E' \subset F \Rightarrow E \subset F$ and $E' \subset F$

$\Rightarrow E \cup E' \subset F \Rightarrow E' \subset F$

Hence $\bar{E} \subset F$

$V_p \cap Y \subset E$ for any $p \in E$

$G \cap Y \subset E$

Thus $E = G \cap Y$

Conversely, let G be open in X and $E = G \cap Y$ then every $p \in E$ has a neighbourhood $V_p \in G$

Thus $V_p \cap Y \subset E$

$\therefore E$ is open relative to Y

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Definition :- Open cover

By an open cover of a set E in a metric space X, we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$

Definition Subcover

A sub collection of $\{G_\alpha\}$ which itself is an open cover is called subcover

Definition compact N'9 Nov 18

A subset K of a metric space X is said to be compact if every open cover of K contains a finite cover (ie) If $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \alpha_2 \ldots \alpha_n$ such that $K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$

Theorem 1·13

April 16 10m Suppose $K \subset Y \subset X$ then K is compact relative to X iff K is compact relative to Y

Proof: Let $K \subset Y \subset X$

Suppose K is compact relative to X

We claim :- K is compact relative to Y

Let $\{V_\alpha\}$ be a collection of sets open relative to Y such that $K \subset \bigcup_\alpha V_\alpha$

By theorem 1·12 suppose $Y \subset X$, A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X

$\therefore$ The set G_α open relative to X such that $V_\alpha = Y \cap G_\alpha \quad \forall \alpha$

Since K is compact relative to X, then we hav

$k \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$, for some choice of finitely many indices $\alpha_1, \alpha_2, \dots \alpha_n$

Since $k \subset Y$ then we have,

$k \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

$\Rightarrow k \subset Y_n \{ G_{\alpha_1} \cup G_{\alpha_2} \dots \cup G_{\alpha_n} \}$

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$\Rightarrow k \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V G_{\alpha_n}$

Hence k is compact relative to Y

We claim :- k is compact relative to y

Let $\{G_\alpha\}$ be a collection of open subsets of X which every k. Let $V_\alpha = Y_n G_\alpha$

then $k \subset V_{\alpha_1} \cup V_{\alpha_2} \cup \dots \cup V_{\alpha_n}$ will hold for some choice of $\alpha_1, \alpha_2, \dots \alpha_n$

Since $V_\alpha \subset G_\alpha$ then we have,

$k \subset V_{\alpha_1} \cup V_{\alpha_2} \dots \cup V_{\alpha_n}$

$\Rightarrow k \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

Hence k is compact relative to X

Theorem 1.14

compact subsets of metric space are closed

Proof:- Let k has compact subset of a metric space X we claim: k is closed

ie) To Prove that the complement of k is open subset of X

Let $p \in X$

then $p \notin k$ and let $q \in k$

Let V_q and W_q be the neighbourhood p and q respectively. of radius less than $\frac{1}{2} d(p,q)$

Since k is compact then there exist a finitely many points $q_1, q_2, \dots q_n$ is k.

We claim H is open

Let $x \in H$

Then there exist a neighbourhood N_i of x with radii, such that $N_i \subset G_i$ $(i=1,2,\ldots n)$

$$\text{let } r = \min(r_1, r_2, \ldots r_n)$$

Let N be the neighbourhood of x radius r

then $N \subset G_i$ for $i=1,2,\cdots,n$

$$\therefore N \subset \bigcap_{i=1}^{n} G_i$$

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ie) $N \subset H$

$\therefore$ H is open

Thus for any finite collection $G_1, G_2, \ldots, G_n$ of open set $\bigcap_{i=1}^{n} G_i$ is open

d) Let $F_1, F_2, \cdots, F_n$ be a collection of closed sets

$$\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} F_i^c \longrightarrow (*)$$

Since F_i is closed $(i=1,2,\ldots n)$

$\therefore F_i^c$ is open

By theorem

For any collection $G_1, G_2, \cdots G_n$ of open sets

$\bigcap_{i=1}^{n} G_i$ is open

By $(*)$ $\left(\bigcup_{i=1}^{n} F_i\right)^c$ is open

$\therefore \bigcup_{i=1}^{n} F_i$ is closed

Thus for ~~any~~ finite collection $F_1, F_2, \ldots, F_n$ of closed sets $\bigcup_{i=1}^{n} F_i$ is closed

Definition:-

If X is a metric space. If $E \subset X$ and if E' denotes the sets of all limit points of E in X then the closure of E is the set $\bar{E} = E \cup E'$

$\Rightarrow I_n \subset G_\alpha$

which is a $\Rightarrow\Leftarrow$ to I_n is not covered by any finite subcollection of $\{G_\alpha\}$

Hence k-cell is compact

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Theorem 1.21

Heine - Borel theorem

If a set E in R^k has one on the following three properties, then it has the other two (a) E is closed and bounded (b) E is compact (c) Every infinite subset of E has a limit point in E

Proof:- $\underline{a \Rightarrow b}$

suppose E is closed and bounded

then $E \subset I$, for some k-cell is I

By theorem 1.20 $[\because E \subset R^k]$

Every k-cell is compact

Hence E is compact

$\underline{b \Rightarrow c}$

suppose E is compact

we Claim:- If every infinite subset of E has a limit point in E

By theorem 1.17

If E is an infinite subset of a compact set k then E has a limit point in k

Hence every infinite subset of E has a limit point in E

$\underline{c \Rightarrow a}$

suppose every infinite subset of E has a limit point in E

we claim:- E is closed and bounded

Suppose E is not bounded

then E contains a points x_n with $|x_n| > n$ $(n = 1, 2, \dots)$ then the set S contains of these point x_n is an infinite

It has no limit point of E

which is a $\Rightarrow\Leftarrow$ to every infinite subset of E has a limit point in E

Theorem : 1.20 Apr - 19

Every k-cell is compact

Proof: Let I be a k-cell

Let I_n consisting of all points $x = (x_1, x_2, \dots, x_n)$ such that $a_j \le x_j \le b_j$ $(1 \le j \le k)$

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$$\text{put } \delta = \left[\sum_{1}^{k} (b_j - a_j)^2 \right]^{1/2}$$

then $|x - y| \le \delta$ if $x, y \in I$

Suppose k-cell is not compact. then there exist an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I Also put $c_j = (a_j \cdot b_j)/2$

The Intervals $[a_j, c_j]$ and $[c_j, b_j]$ then the determinds 2^k, k-cells Q_i whose union is I

Atleast one of these sets Q_i call it I_1

Its cannot be covered by any finite subcollection $\{G_\alpha\}$ we next sub divide I_1 and continue the process, we get

a) $I \supset I_1 \supset I_2 \dots$

b) I_n is not covered by any finite subcollection of $\{G_\alpha\}$

c) If $x \in I_n$ and $y \in I_n$ then $|x - y| \le 2^{-n} \cdot$

By (a) and theorem 1.19,

Let k be the positive integers. If $\{I_n\}$ is a sequence of k-cell such that $I_n \supset I_{n+1}$ $(n = 1, 2, \dots)$ then $\bigcap_1^\infty I_n$ is non-empty

Then there exist a point x^* which lies in every I_n ie) $x^* \in G_\alpha$ for some α

Since G_α is open. then there exist $r > 0$ such that $|y - x^*| < r \Rightarrow y \in G_\alpha$

If n is so large that $2^{-n} \delta < r$

Now by (iii) V_{n+1} satisfies our Induction hypotheses and the construct can proceed

Put $K_n = \overline{V_n} \cap P$

since $\overline{V_n}$ is closed and bounded

ie). $\overline{V_n}$ is compact

Since $x_n \notin K_{n+1}$ no point of P lies in $\bigcap_1^\infty K_n$

ie) $K_n \subset P \Rightarrow \bigcap_1^\infty K_n$ is empty.

By (iii) and $K_n \supset K_{n+1}$ By (i)

Which is contradiction sets such that 26

$K_n \supset K_{n+1} \quad (n = 1, 2, \cdots)$

then $\bigcap_1^\infty K_n$ is non-empty.

Hence P is uncountable

Note:

Every Interval $[a, b]$ is uncountable In particular the set of all real number is uncountable.

Definition Cantor set

Let E_0 be the Interval $[0,1]$ Remove the segment $(1/3, 2/3)$ and let E_1 be the union of the intervals $[0, 1/3]$ $[2/3, 1]$ Removed the middle thirds of these intervals and let E_2 be the union of these intervals and let E_2 be the union of these intervals $[0, 1/9]$ $[2/9, 3/9]$ $[6/9, 7/9]$ $[8/9, 1]$ ... then we obtain a sequences of compa set E_n such that

a) $E_1 \supset E_2 \supset \cdots$

b) E_n is the union of 2^n intervals each of length 3^{-n}. The set $P = \bigcap_{n=1}^\infty E_n$ is called cantor set

Definition :-

E is perfect - If E is closed and if every point of E is a limit point of E.

Theorem 1.11

Let E be a non-empty set of real numbers which is bounded above. Let $y = \sup E$. then $y \in \bar{E}$ Hence $y \in E$ if E is closed

Proof :- Let E be a non-empty set of real numbers

Let $y \in \sup E$ [E which is bounded above]

Calim :- $y \in E$

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Assume that $y \notin E$

∴ For any $h > 0$, then there exist a point $x \in E$ such that

$y-h < x < y$ Hence $y-h$ is an upper bound of E

Which is $\Rightarrow \Leftarrow$

[$|x-h| < h$ $\Rightarrow -h < x-y < h$ $\Rightarrow y-h < x < y+h$ $\Rightarrow y-h < x < h$]

∴ y is a limit point of E

Hence $y \in \bar{E}$ [∴ $y \in E$ and $y \in E'$]

Definition:-

Let x be a metric space and Let $E \subset Y \subset X$. E is open relative to Y. If to each $p \in E$ then there exist $r > 0$ such that $d(p, q) < r$

AP. 19

uq. 5m Theorem 1.12

suppose $Y \subset X$ A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X

proof :- Suppose $Y \subset X$ and E is open relative to Y

then $E \subset Y \subset X$

To each $p \in E$, then there exist $r_p > 0$

such that $d(p, q) < r_p$ $q \in Y \Rightarrow q \in E$

Let V_p be the set of all $q \in X$ such that $d(p, q) < r_p$ Let $G = \bigcup_{p \in E} V_p$

Then G is open set of X [by theorem 1.5 & 1.9 (a)]

We claim - $E = G \cap Y$

[Every neighbourhood is an open set and for any collection $\{G_\alpha\}$ of opensets $\bigcup_\alpha G_\alpha$ is open].

Since $p \in V_p$ $\forall p \in E$ it is clear that

$E \subset G \cap Y$ By our choice of V_p we have where I is a k-cell

By theorem 1·20

Every k-cell is compact

$\therefore$ I is compact

By theorem 1·21 For $E \subset R^k$ the following are equivalent

a) E is closed and bounded

(b) E is compact

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(c) Every infinite subset of E has a limit point in E

$\therefore$ E has a limit point in R^k.

Perfect sets.

Definition: countable.

let J denote the set of all positive integers. A set A is said to be countable if, there exist a one-to one map from A to J

Theorem : 1·23

Let P be a non empty perfect set in R^k then P is uncountable

Proof: Let P be a non-empty perfect set in R^k

Since P has a limit point and P must be infinite

we claim:- , P is uncountable

suppose P is countable say $P = \{x_1, x_2, \ldots x_n\}$

let V_1 be any neighbourhood of x_1. If V_1 consists of all $y \in R^k$ such that $|y - x_1| < r$

The closure of $\overline{V_1}$ of x_1 is the set of all $y \in R^k$ such that $|y - x_1| \leq r$

Suppose V_n has been constructed so that $V_n \cap P$ is non-empty.

Since every point p is a limit point of P and neighbourhood V_{n+1} such that

i) $\overline{V_{n+1}} \subset V_n$

ii) $x_n \notin \overline{V_{n+1}}$

iii) $V_{n+1} \cap P$ is non-empty.

Let p be a limit point of the set E

claim: Every neighbourhood of p contains infinitely many point of E

suppose there is a neighbourhood of p which contains only a finite number of point of E

Let $q_1, q_2 \ldots q_n$ be those points of $N \cap E$ which is distinct from p

put $r = \min (1 \leq m \leq n)$, $d(p, q_m)$

since $p \neq q_i$, $d(p, q_i) > 0$ for all $i = 1, 2 \ldots n$

$\therefore r > 0$

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ie) The neighbourhood $N_r(p)$ contains no points q of E such that $q \neq p$

p is not a limit point of E

which is contradiction to our assumption

Hence every neighbourhood of p contains infinitely many point of E

Corollary:

A finite point set has no limit point

Theorem 1.7

Let $\{E_\alpha\}$ be a finite (or) infinite collection of sets E_α then $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\infty (E_\alpha)^c$

Proof: Let $x \in \left(\bigcup_\alpha E_\alpha\right)^c$

$\therefore x \notin \bigcup_\alpha E_\alpha$

ie) $x \notin E_\alpha$ for any α

$\therefore x \in E_\alpha^c$ For every α

$\therefore x \in \bigcap_\alpha (E_\alpha)^c$

$\therefore \left(\bigcup_\alpha E_\alpha\right)^c \subset \bigcap_\alpha (E_\alpha)^c \rightarrow (1)$

Now, Let $x \in \bigcap_\alpha E_\alpha^c$

ie) $x \in E_\alpha^c$ For every α

b) For any collection $\{F_\alpha\}$ of closed sets $\bigcap_{\alpha} F_\alpha$ is closed

c) For any finite collection $G_1, G_2, \dots G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open

d) For any finite collection $F_1, F_2, \dots F_n$ of closed sets $\bigcup_{i=1}^{n} F_i$ is closed

(13)

Proof: Let $\{G_\alpha\}$ be a collection of open sets

Let $G = \bigcup_{\alpha} G_\alpha$

Let $x \in G$

then $x \in G_\alpha$ for some α

since x is an interior point of G

Hence for any collection $\{G_\alpha\}$ of open sets $\bigcup_{\alpha} G_\alpha$ is open

b) Let $\{F_\alpha\}$ be a collection of closed sets

By theorem 1.7

$$\left(\bigcap_{\infty} F_\alpha\right)^c = \bigcup_{\alpha} F_\alpha^c \longrightarrow \circledast$$

Since F_α is closed, ie) F_α^c is open

By theorem

For any collection $\{G_\alpha\}$ of open sets

ie) $\bigcup_{\alpha} F_\alpha^c$ is open

$\therefore \bigcup_{\alpha} F_\alpha^c$ is open

since $\left(\bigcap_{\infty} F_\alpha\right)^c$ is open [$\because$ by $*$]

Hence $\bigcap_{\infty} F_\alpha$ is closed

Thus for any collection $\{F_\alpha\}$ of closed sets $\bigcap_{\infty} F_\alpha$ is closed

c) Let $G_1, G_2, \dots G_n$ be a collection of open sets

Let $H = \bigcap_{i=1}^{n} G_i$

c) Now, $d(x,z) = |x-z|$

$$\leq |x-y| + |y-z|$$

$$\leq d(x,y) + d(y,z)$$

$\therefore d(x,z) \leq d(x,y) + d(y,z)$

$\therefore d(x,y)$ satisfies the three conditions of the metric $\therefore d(x,y)$ is metric

Remark: $(a,b) = \{x \in R : a < x < b\}$

$[a,b] = \{x \in R : a \leq x \leq b\}$

$[a,b) = \{x \in R : a \leq x < b\}$

$(a,b] = \{x \in R : a < x \leq b\}$

k-cell

Definition

If $a_i < b_i$ for $i = 1, 2, \ldots k$ the set of all points $x = (x_1, x_2, \cdots x_k)$ in R^k whose co-ordinates satisfy the inequalities $a_i \leq x \leq b_i$ $(1 \leq i \leq k)$ is called a k-cell

Remark: 1-cell is an interval

2-cell is an rectangle

Definition

Open Ball

If $x \in R^k$ and $r > 0$ then open ball B with enter at x and radius r is defined by

$$B(x,r) = \{y \in R^k : |y - x| < r\}$$

Definition: Closed Ball

If $x \in R^k$ and $r > 0$ the closed ball B with Center at x and radius r is defined by $B[x,r] = \{y \in R^k : |y - x| \leq r\}$

Definition convex

A set $E \subset R^k$ is said to be convex if $\lambda x + (1-\lambda) y \in E$ Where $x \in E$, $y \in E$ and $0 < \lambda <$

vii) The complement of E (denoted by E^c), is the set of all points $p \in X$ such that $p \notin E$

viii) E is perfect if E is closed and if every point of E is a limit point of E

xi) E is bounded if there is a real number m and a point $q \in X$ such that $d(p,q) < m$ for all $p \in E$ (10)

(x) E is called dense set in X if every point of X is a limit point of E (or) a point of E or both

Theorem 1.5

2m Every neighbourhood is an open set

proof: Let $E = N_r(P)$ be a neighbourhood

Let $q \in E$ $[\because q \in N_r(P) \Rightarrow d(P,q) < r$

$d(P,q) = r - h]$

Then there exist a real number h such that $d(p,q) = r-h$ for all points s such that $d(P,s) < h$

Now, $d(P,s) < d(P,q) + d(q,s)$

$< r - h + h$

Hence $d(P,s) < r$

$\therefore s \in N_r(P)$

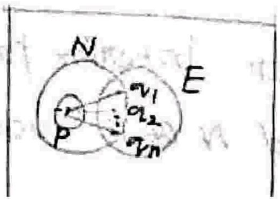
ie) q is an interior point of E

Thus E is open.

Theorem: 1.6 (m)(x)

If P is a limit point of a set E then every neighbourhood of P contains infinitely many points of E

Proof:-



UNIT - I

Basic Topology

Finite, countable and uncountable sets

1. Function From A to B

Let A and B be non-empty sets each elements x of A associates with an element of B in some manner then 'f' is said to be Function From A to B (or) mapping of A into B. then elements of B is denoted by $f(x)$. the set of all values of f is called the range of 'f'

2. Image:- Let A and B be two non-empty set and Let 'f' be a mapping of A into B if $E \subset A$ then the set of all elements $f(x)$ for $x \in E$ is said to be image of E under 'f' denoted by $f(E)$

3. Inverse Image:- Let A and B be two non-empty sets and let 'f' be a mapping of A into B. If $E \subset B$ then inverse image of E under 'f' is denoted by $f^{-1}(E)$ and defined by $f^{-1}(E) = \{x \in A, f(x) \in E\}$

If $y \in B$ $f^{-1}(y)$ is the set of all $x \in A$ such that $f(x) = y$

4. Injective:- Let A and B be two non-empty sets the Function $f: A \to B$ is said to be one to one Function. if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$

then $E_1 \cup E_2 = \{1, 2, 3, 4\}$ and

$E_1 \cap E_2 = \{2, 3\}$

2) let $A = \{x \in R, 0 \le x \le 1\}$ For every $x \in A$

$E_x = \{y \in R : 0 < y < x\}$

then i) $E_x \subset E_z$ iff $0 < x \le z < 1$

ii) $\bigcup_{x \in A} E_x = E_1$ 5

iii) $\bigcap_{x \in A} E_x$ is empty.

Theorem 1.2

Let $\{E_n\}$, $n = 1, 2, 3, \dots$ be a sequence of countable sets and put $S = \bigcup_{n=1}^{\infty} E_n$ then S is countable.

Proof:

Let Every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, \dots$ Consider these elements as

$$\begin{array}{cccc} x_{11} \rightarrow & x_{12} & x_{13} \rightarrow & x_{14} \dots \\ x_{21} & x_{22} & x_{23} & x_{24} \dots \\ x_{31} & x_{32} & x_{33} & x_{34} \dots \\ x_{41} & x_{42} & x_{43} & x_{44} \dots \\ \dots & \dots & \dots & \dots \end{array}$$

This infinite array contains all element of S. Arrange these elements in a sequence.

$$x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}$$
$$x_{41}, x_{32}, x_{23}, x_{14}, \dots \quad \rightarrow (1)$$

If any two of the sets E_n have elements in common. These will appear more than once in (1)

Hence $S \sim T$ where T is the set of all the integer

$\therefore S$ is countable

Corollary:

Suppose A is atmost countable and For every $\alpha \in A$, B_α is atmost countable

put $T = \bigcup_{\alpha \in A} B_\alpha$

Then T is atmost countable

Note: * Countable sets are called enumerable (or) denumerable

* If $A \sim B$, we also say that A and B have the same cardinality number or A and B can be put in one to one correspondence

Ex: Prove that the set of all integer 'z' is countable

Proof:- Claim: $Z \sim J$

where $Z = \{0, \pm 1, \pm 2 \pm, \dots\}$

and $J = \{1, 2, \dots\}$ we can define the function 'f' from J to z by

$$f(n) = \begin{cases} n/2 & , \text{ if } n \text{ is even} \\ \frac{-(n-1)}{2} & \text{ if } n \text{ is odd} \end{cases}$$

Then 'f' has a one to one correspondence between J and z. Hence z is countable

Remark: * A finite set cannot be equivalent to one of its proper subsets

A is finite if 'A' is equivalent to one of its proper subsets

9. The terms of the sequence:-

A function 'f' defined on the sets 'J' of all positive integer by $f(n) = x_n$ for all $n \in J$. then the sequence 'f' denoted by the symbol sequence $\{x_n\}$ or $x_1, x_2, x_3, \dots$ The value of f, x_n are called the terms of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$ Then $\{x_n\}$ is said to be a sequence in A (or) a sequence of element of A

Note: The terms $x_1, x_2, x_3, \dots$ of a sequence need not be distinct

Theorem 1·1

Every infinite subset of a countable set A is countable

Proof: Let $E \subset A$ and E is infinite

Arrange the elements x of A in a sequence $\{x_n\}$

Remark
open ball are convex. Let $x \in B(x,r)$ $z \in B(x,r)$ and $0<\lambda<1$ if $y \in B(x,r) \Rightarrow |y-x|<r$
if $z \in B(x,r) \Rightarrow |z-x|<r$

Now,

$$|\lambda y+(1-\lambda)z-x| = |\lambda y+\lambda x-\lambda x+(1-\lambda)z-x|$$
$$= |\lambda(y-x)+(1-\lambda)(z-x)|$$
$$\le \lambda|y-x|+(1-\lambda)|z-x|$$
$$< \lambda r+(1-\lambda)r < \cancel{\lambda r}-\cancel{\lambda r}+r$$
$$< r$$

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$$|\lambda y+(1-\lambda)z-x|<r$$
$$\Rightarrow \lambda y+(1-\lambda)z \in B(x,r)$$

1) open balls are convex
2) closed balls are convex
3) k-cells are convex

Some Definitions

i) A neighbourhood of a point p is a set $N_r(p)$ defined by $N_r(p)=\{q : d(p,q)<r\}$. The number 'r' is called the radius of $N_r(p)$

ii) A point 'p' is a limit point of the set E. If every neighbourhood of p contains a point E. $q \ne p$ such that $q \in E$

iii) If $p \in E$ and p' is not a limit point of E then p is called an isolated point of E

iv) E is closed if every limit point of E is a point of E

v) E is open if every point of E is an interior point of E

vi) A point p is an interior point of E if there is a neighbourhood N of p such that $N \subset E$

For all $x_1 \in A$, $x_2 \in$ ..

5 Surjective:

Let A and B be two non-empty sets the Function $f: A \to B$ is said to be onto Function if $f(A) = B$

i.e) Every element of B has a pre-image in A.

6 Bijection:

If Function $f: A \to B$ is both one-one and onto then f is called a bijection

7 Equivalent:

If there exists a 1-1 mapping of A onto B then we say that A and B are equivalent We write $A \sim B$

Remark: The relation '$\sim$' has the Following Property

Reflexive: $A \sim A$

Symmetric: If $A \sim B$ then $B \sim A$

Transitive: If $A \sim B$ and $B \sim C$ then $A \sim C$

Any reflexive with these three properities is called an equivalence relation

Countable and uncountable:

For any positive integer n. Let $J_n = \{1, 2, \ldots\}$ for any set A

* A is finite if $A \sim J_n$, for some n
* A is infinite if A is not finite
* A is countable if $A \sim J$
* A is uncountable if A is neither finite nor countable
* A is atmost countable if A is finite or uncountable

Theorem 1·24
smw

p is Perfect (or) Cantor is Perfect

Proof :- To show that p is perfect

It is enough to show that p contains no isolated points

Let $x \in p$

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Let δ be any segment containing x

let I_n be the interval of E_n which contains x

Now, we choose n large enough to so that $I_n \subset S$

Let x_n be a end point of I_n such that $x_n \neq x$

It follows from the construction p such that $x_n \in p$

Hence x is a limit point of p

Hence p is perfect

Hence E has atleast one limit point

Theorem 1.18

If $\{I_n\}$ is a sequence of intervals in R^1 such that $I_n \supset I_{n+1}$ $(n=1,2,\dots)$ then $\cap I_n$ is non-empty.

Proof: Let $I_n = [a_n, b_n]$ (21)

let E be the set of all a_n

then E is non-empty and bounded above

let x be the sup of E

If m and n are positive integers then $a_n \leq a_{m+n} \leq b_{m+n} \leq b_m$

so that, $x \leq b_m$ foreach m

Since it is obvious that $a_m \leq x$

$\therefore x \in I_m$ for $m = 1, 2, \dots n$

Hence $\bigcap_1^\infty I_n$ is non-empty.

Theorem 1.19

let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n=1,2,\dots)$ then $\bigcap_1^\infty I_n$ is non-empty.

Proof: Let k be a positive integer

Let $\{I_n\}$ be a sequence of k-cells such that $I_n \supset I_{n+1}$ where $(n=1,2,\dots)$

We define $I_{n,j} = [a_{nj}, b_{nj}]$

By theorem 1.1

If I_n is a sequence of an interval in R^1 such that $I_n \supset I_{n+1}$ where $n=1,2,\dots$ then $\bigcap_1^\infty I_n$ is non-empty.

$\therefore$ For each j the sequence $\{I_{n,j}\}$ satisfy.

Hence there exist a real number x_j^* $(1 \leq j \leq k)$ such that $a_{n,j} \leq x_j^* \leq b_{n,j}$ $(1 \leq j \leq k)$ $(n=1,2,\dots)$ Now we take $x^\infty = (x_1^\infty, x_2^\infty, \dots, x_n^\infty)$

$\therefore x^\infty \in I_n$ for $n=1,2,\dots$

Hence $\bigcap_1^\infty I_n$ is non-empty.

proof: let 'E' be a countable subset of A

let E consist of the sequence $S_1\ S_2\ S_3 \dots$

We construct a sequence 's' as follows

If the n^{th} digit in s_n is 1 then n^{th} digit of s is 1

Then the sequence 'S' differs from every number of E is atleast one place

Hence $S \notin E$, but $S \in A$

$\therefore$ E is a proper subset of 'A'

Hence every countable subset of 'A' is a proper subset of A

Hence A is uncountable.

5m Nov'11 2m **Metric space**: 2m

A non-empty set X is said to be metric space if a function $d: X \to X \to R$ satisfies the following conditions

i) $d(p,q)=0$ if $p \neq q$

$d(p,p)=0$

ii) $d(p,q) = d(q,p)$

iii) $d(p,q) \leq d(p,r)+d(r,q)$ for every $r \in X$

A real number $d(p,q)$ is called the distance from p to q and the Function d is called a distance Function (or) metric

Example: Prove that $d(x,y) = |x-y|$ where $x,y \in R^k$ is metric

Proof: Clearly, $d(x,y) = |x-y| > 0$ if $x \neq y$

suppose if $x=y$

$d(x,x) = |x-x|=0$

$\therefore d(x,x)=0$

b) $d(x,y) = |x-y| = |y-x| = d(y,x)$

$\therefore d(x,y) = d(y,x)$

such that $k \subset W_{q_1} \cup W_{q_2} \cup \dots \cup W_{q_n} = W$

If $V = V_{q_1} \cap V_{q_2} \dots \cap V_{q_n}$

Then V is a neighbourhood of p which does not intersect W

Hence $V \subset k^c$

$\therefore$ p is an interior point of k^c

$\therefore$ k^c is an open subset (19)

Hence k is closed.

Theorem 1:15

Closed subset of a compact sets are compact

proof: suppose $F \subset k \subset X$

Let F be a closed set and let k be a compact

we claim: F is compact

Let $\{V_\alpha\}$ be an open cover of F

If F^c is disjointed to $\{V_\alpha\}$ then we obtained an open cover Ω of k

since k is compact, then there is a finite sub collection ϕ of Ω which covers k.

ie) $\phi = k$

If F^c is a member of ϕ then we may remove it form ϕ and still retain on open cover of F

A finite sets collection of $\{V_\alpha\}$ cover F $\therefore$ F is compact

Note:- If F is closed and k is compact then $F \cap k$ is compact.

Theorem 1.16

If $\{k_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite sub collection of $\{k_\alpha\}$ is non-empty then $\bigcap_{\alpha} k_\alpha$ is non-empty.

Proof: Let k_1 be a member $\{k_\alpha\}$

Let $G_\alpha = k_\alpha^c$

Assume that, no point k_1 belongs to every k_α then the sets G_α form an open cover of k_1

Since k_1 is compact then there exist a finitely many indices $\alpha_1, \alpha_2 \cdots \alpha_n$

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$$k_1 \subset \cup G\alpha_1 \cup G\alpha_2 \cup \cdots \cup G\alpha_n$$

But this means that $k\alpha_1 \cap k\alpha_2 \cap \cdots \cap k\alpha_n$ which is $\Rightarrow\Leftarrow$ to our assumption

Hence $\bigcap_\alpha K_\alpha$ is non-empty.

Result:-

If $\{k_n\}$ is a sequence of non-empty compact sets such that $k_n \supset k_{n+1}$ $(n = 1, 2, 3, \ldots)$ then $\bigcap_{1}^{\infty} k_n$ is non-empty.

Theorem 1.17.

If E is an infinite subset of a compact k, then E has a limit point in k

Proof:- Let k be a compact set

let E be infinite subset of k

we claim : E has a limit point in k

ie) To prove, E has atleast one limit point

suppose E has no limit in k

Let $q \in k$

Since q is not a limit point of k

then there exist a neighbourhood. V_q such that $V_q \cap (E - \{q\}) = \phi$

$$\therefore V_q \cap E = \begin{cases} \{q\} & \text{if } q \in E \\ \phi & \text{if } q \notin E \end{cases}$$

Now, $\{V_q \mid q \in k\}$ is an open cover for k

Also, each V_q covers atmost one point of infinite ~~subcover~~ set E

Hence this open cover cannot have a finite subcover which is $\Rightarrow\Leftarrow$

[$\because$ k is compact]

ie) $x \notin E_\alpha$ For any α

ie) $x \notin \bigcup_\alpha E_\alpha$

ie) $x \in (\bigcup_\alpha E_\alpha)^c$

$\therefore \bigcap_\alpha E_\alpha^c \subset (\bigcup_\alpha E_\alpha)^c \rightarrow (2)$

From (1) and (2) we get,

$$(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha)^c$$

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5m 2m. Theorem : 1.8 NOV-18

UQ The set E is open iff its complement is closed (E^c is closed)

Proof: Let X be a metric space and $E \subset X$

suppose E^c is closed

We claim :- E is open

Let $x \in E$, then $x \notin E^c$

$\therefore x$ is not a limit point of E^c

Hence there exist a neighbourhood N of x such that $E^c \cap N$ is empty. ie) $N \subset E$

$\therefore x$ is an interior point of E

Hence E is open

Conversely, suppose E is open

Claim : E^c is closed

Let x be a limit point of E^c

then every neighbourhood of x contains a point of E^c $\therefore x$ is not an interior point of E

Since E is open

ie) $x \in E^c$

Hence E^c is closed.

Theorem : 1.9

a) For any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ is open

Now, suppose every infinite subset of E has a limit points in E

we claim :- E is closed

Suppose E is not closed

then there exist a point $x_0 \in R^k$ which is a limit point of E but not a point in E

For $n = 1, 2, \cdots$ there are some points $x_n \in E$ such that $|x_n - x_0| < 1/n$

Let S be the set of these points x_n then S is finite otherwise $|x_n - x_0|$ would has a positive value for infinitely many values n).

$\therefore S$ has x_0 as a limit point in R^k for if $y \in R^k$, $y \neq x_0$

$$\begin{aligned} \text{then } |x_n - y| &= |x_n - x_0 + x_0 - y| \\ &\geq |x_0 - y| - |x_0 - x_n| \\ &\geq |x_0 - y| - |x_n - x_0| \\ &\geq |x_0 - y| - 1/n \\ &\geq 1/2 \, |x_0 - y| \end{aligned}$$

For all by finitely many n

Hence y is not a limit point S

Thus S has no limit point in E

which is $\Rightarrow\Leftarrow$ to our assumption

$\therefore E$ is closed

From ① & ②, we have,

E is closed and bounded

Theorem 1.22 weierstass theorem 19 NOV 15

Every bounded infinite subset of R^k has a limit point in R^k

Proof :- Let E be a bounded infinite subset of R^k we claim : E has a limit point in R

Since E is bounded

(1)

Unit - II

convergent sequence:

A sequence $\{P_n\}$ is a metric space is said to convergent. if there is a point $P \in X$ with the following property. for every $\varepsilon > 0$ there is an integer $N \ni : m \geq N$ implies that $d(P_n, P) < \varepsilon$ [Here d denotes the distance in X].

In this case we also say that $\{P_n\}$ converges to P or ~~these~~ that P is the limit of $\{P_n\}$ and we write $\{P_n\} \to P$.

or $\lim_{n \to \infty} P_n = P$.

If $\{P_n\}$ doesnot convergent it is said to be divergent.

Bounded:

The set of all points $P_n (n = 1, 2 \ldots)$ is the range of $\{P_n\}$. The range of a sequence may be a finite set or it may be a infinite set. The sequence $\{P_n\}$ is said to be bounded if its range is bounded.

unit - III

power series

V.Q

Defn:

Given a sequence $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series.

note:

when numbers c_n are called the coefficient of the series, z is a complex numbers.

V.Q

Thm: 3.39

Given the power series $\sum c_n z^n$. put $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|c_n|}$, $R = 1/\alpha$ then $\sum c_n z^n$ converges $|z| < R$ and if deverges $|z| > R$.

PF:

$$\lim_{n\to\infty} \sup \sqrt[n]{|c_n|}$$

we take $c_n = c_n z^n$

$$= \lim_{n\to\infty} \sup \sqrt[n]{|c_n \cdot z^n|}$$

$$= \lim_{n\to\infty} \sup \sqrt[n]{|c_n||z^n|}$$

unit - IV

Limit of a function:

Let X and Y be metric space. Suppose $E \subset X$, f maps E into Y and p is a limit point of E. we write $f(x) \to q$ as $x \to p$ or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property. For every $\varepsilon > 0$ there exist $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta$$

U.Q 5m Thm:

Let X and Y be metric space. and $E \subset X$. f maps E into Y and p is a limit point of E. Then.

$$\lim_{x \to p} f(x) = q$$

Left limit:

A function f is said to have l as the left at $x=a$ if given $\epsilon > 0$ there exists $\delta > 0$ such that $a-\delta < x < a \Rightarrow |f(x) - l| < \epsilon$ and we write $\lim_{x \to a^-} f(x) = l$ denoted by $f(a^-)$

Unit - V

Definition:

(1) Let f be defined on $[a, b]$ for any $x \in [a, b]$ from the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, \ t \neq x)$$

and define

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$